confined to one fast disappearing pool, would be observed when dotting the ground over an extent perhaps of an acre or more. Seen thus, immediately after rain, and not previously noticed, the inference is not so strange that they came to the earth with the rain, or that there had been a shower of toads as well as of water.

Trenton, N. J., April 7, 1904.

## EXPANSIONS OF ALGEBRAIC FUNCTIONS AT SINGULAR POINTS.

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(Read April 7, 1904.)

## I. Introduction.

An algebraic equation $F(x, y)=0$ of degree $n$ in $y$ defines $y$ as an $n$-valued algebraic function of $x$. When these $n$ values of $y$ are all distinct for a given value of $x$, that value of $x$ is called a regular point of the algebraic function, and the $n$ branches of the function are extended by applying the law of the continuity of each branch.

In curve tracing $x$ and $y$ are real variables and only the real branches of the function are used. Real values of $x$ and $y$ which
 tiple points of the curve which represents the equation $F(x, y)=0$. If $x=a, y=b$ is a multiple point of this curve, the behavior of the curve at the multiple point is determined from the expansions of $y-b$ in terms of $x-a$. Inasmuch as the transformations

$$
x=x_{1}+a, y=y_{1}+b
$$

transfer the origin to the multiple point, the multiple point will always be taken at the origin.

An algebraic equation between complex variables $F(w, z)=0$ of degree $n$ in $w$ defines $w$ as an $n$-valued algebraic function of $z$. Values of $w$ and $z$ which satisfy the equations $F(w, z)=0$ and $\frac{d}{d w} F(u, z)=0$, determine branch points of the algebraic function, that is points where several branches of the function meet. The behavior of the function at a branch point is determined from the expansions of the function at the branch point.

The multiple points in curve tracing and the branch points in algebraic equations between complex variables are grouped as the singular points of algebraic functions.

## II. Historical.

The problem of the expansion of algebraic functions at singular points dates back to Newton. Newton's "parallelogram method" determines the first term of the expansions as follows. The equation transformed to the singular point as origin becomes

$$
\sum_{a_{m \mathrm{~m}}} a^{\mathrm{m}} y^{\mathrm{n}}=0 .
$$

Locate on squared paper to rectangular axes the points ( $m, n$ ) whose coördinates are the exponents of $x$ and $y$ in the various terms of the transformed equation. Connect by successive straight lines, forming a broken line convex toward the origin, the points nearest the origin. The sums of the terms of $\Sigma a_{m n} x^{\mathrm{m}} y^{\mathrm{n}}=0$, for which the points ( $m, n$ ) are located on the same straight line, when equated to zero form equations which determine the first terms of the expansions at the singular point.

Puiseux in his classical "Memoir" on Algebraic Functions," Liouville's Journal, t. XV, 1850, used Newton's parallelogram method and studied in detail the nature of the expansions of algebraic functions. Puiseux's Memoir is made the basis of Briot and Bouquet's "Elliptic Functions," and indeed is almost universally used in the study of algebraic functions.

Nöther's method in Annalen, IX, 1876, is representative of the more recent methods of expansion of algebraic functions. By successive quadratic transformations the singular point becomes a regular point, and from the expansions at this regular point the expansions at the original singular point are obtained by reversing the quadratic transformations and the reversion of series.

In the present paper an analytic method is presented which determines not only the first terms of the expansions but also the successive approximations of the several expansions. The method of expansion used is that application of Maclaurin's series which the author employed to compute all the roots of numerical equations and which is published in Vol. XLII of the Proceedings of the American Philosophical Society.

## III. A New Method of Expansion.

For convenience of description the exponents in the equation to which the method is applied are assumed numerical.

Suppose the algebraic equation when the singular point is taken as:origin to have the form
(1) $G x^{6} y^{14}+F x^{3} y^{13}+H x^{10} y^{12}+J x^{3} y^{11}+K x^{8} y^{10}+E y^{10}$

$$
+I x^{14} y^{7}+\underline{D x^{3} y^{5}}+L x^{9} y^{4}+\underline{C x^{7} y^{2}+B x^{10} y+A x^{13}=0, ~}
$$

where the terms are arranged according to the descending powers of $y$. In this equation $y$ has fourteen branches, which are to be separated at the singular point by expanding $y$ as a function of $x$.

The terms of equation ( 1 ) to be underscored are determined by the method used for this purpose in the paper on the "Solution of Equations," and which is adapted to the present case as follows. If $L x^{m_{1}} y^{n_{1}}, M x^{m_{2}} y^{n_{2}}, N x^{m_{3}} y^{n_{3}}$ are any three terms of equation ( I ), the value of

$$
\begin{aligned}
& \text { (2) limit } \frac{M_{n_{1}-n_{3}}}{x=0} \frac{x^{m_{2}}\left(n_{1}-n_{3}\right)}{L^{n_{2}-n_{3}} \Lambda^{n_{1}-n_{2}}} \frac{\left.x^{m_{1}}\left(n_{2}-n_{3}\right)^{n_{3}}\right) x^{m_{3}}\left(n_{1}-n_{2}\right)}{}
\end{aligned}
$$

is zero, finite, or infinite. It is at once seen that this iimit is zero, finite, or infinite, according as $m_{2}\left(n_{1}-n_{3}\right)$ is greater than, equal to, or less than $m_{1}\left(n_{3}-n_{3}\right)+m_{3}\left(n_{1}-n_{2}\right)$.

The underscored terms of equation ( r ) are all the terms which satisfy the following condition. The limit (2) for any three consecutive underscored single terms is infinite. If a group of terms is underscored as a single term, the limit (2) is finite for all the terms of this group, and the limit is infinite for the first term of the group and the next preceding underscored term, the limit is also infinite for the last term of the group and the next succeeding underscored term.

We now proceed to underscore the terms of equation (x) to satisfy this condition.

Underscore the first term of (1), and determine the limit (2) for the first three terms of ( 1 ). Since the limit is infinite, underscore the sccond term of ( I ) temporarily.

Determine the linit (2) for the terms 2, 3, 4. The limit is zero and term 3 is not underscored. Next determine the limit (2) for the terms $2,4,5$. The limit is infinite and term 4 is temporarily, term 2 permanently underscored.

The limit for terms 4, 5, 6 is zero, the limit for terms $\mathbf{x}, 2,6$ is infinite. Hence term;4 does not remain underscored, and term 6 is temporarily underscored.

The limit for the terms 2, 6, 7 is infinite, the limit for terms 6, 7 , 8 is zero, the limit for the terms $2,7,8$ is infinite. Hence term 6 permanently underscored, term 7 is not underscored, and term 8 is temporarily underscored.

The limit for terms $6,8,9$ is infinite, for terms $8,9,10$ zero, for terms 6,8 , 10 infinite. Hence term 8 is permanently underscored, term 9 is not underscored, and term ro is temporarily underscored.

The limit for terms 8, 10 , 1 I is infinite, hence term 10 is permanently underscored.

The limit for terms ro, 11,12 is finite, and these three terms are underscored as one term.

The several equations formed by retaining in equation (x) in succession only consecutive underscored terms, if these terms are single, and if a group of terms is underscored by retaining only the group of terms, the first term of the group and the next preceding underscored term, and the last term of the group and the next succeeding underscored term, will determine the first approximations of the fourteen branches of the function.

These equations are

$$
\begin{aligned}
& \text { a) } G x^{3} y+F=0, \quad \text { b) } F x^{3} y^{3}+E=0, \quad \text { c) } E y^{5}+D x^{3}=0, \\
& \text { d) } D y^{3}+C x^{4}=0, \quad \text { e) } C y^{2}+B x^{3} y+A x^{6}=0,
\end{aligned}
$$

and the fourteen first approximations are

$$
\begin{aligned}
& \text { a) } y=-\frac{F}{\dot{G}} \frac{1}{x^{3}} \text {, b) } y=\left(-\frac{E}{F}\right)^{\frac{3}{x}} \frac{1}{x}, \text { c) } y=\left(-\frac{D}{E}\right)^{\frac{b}{1}} x^{\frac{1}{2}}, \\
& \text { d) } y=\left(-\frac{C}{D}\right)^{\frac{3}{3}} x^{\frac{3}{3}}, \text { e) } y=\frac{-B \pm \sqrt{B^{2}-4 A C}}{2 C} x^{3} .
\end{aligned}
$$

Of these fourteen branches the separate branch $a$ ) and the three separate branches $b$ ) go through infinity when $x=0$. The cycle of five branches $c$ ), the cycle of three branches $d$ ), and the two separate branches e) constitute the ten branches which meet at the singular point.

If a factor $t$ is introduced in succession into all the terms of equation ( 1 ) except the terms used to determine the first approximation of a branch of the function, the successive approximations of this branch are determined by developing $y$ in ascending powers of $t, x$ considered constant, by Maclaurin's Series and making $t$ unity in the result.

## IV. Behavior of Branches of Algebraic Functions.

If the algebraic equation takes the form $\Sigma a_{\mathrm{mx}} x^{\mathrm{m}} y^{\mathrm{n}}=0$ when a regular point is taken as origin the method of expansion determines $n$ separate branches of the function.

If the origin is a singular point of $\Sigma a_{\mathrm{mn}} x^{m} y^{\mathrm{n}}=0$ the behavior of the several branches is determined as follows.
a) If the first approximation is independent of $x$ or contains a negative power of $x$, the corresponding branches are either finite or infinite and consequently these branches do not go through the singular point.
b) To each pair of consecutive underscored terms in which the exponents of $y$ differ by unity there corresponds a separate branch of the function through the singular point.
c) To each pair of consecutive underscored terms in which the exponents of $y$ differ by more than unity there corresponds a separate cycle of branches hanging together at the singular point, provided the exponents of $x$ and $y$ in the equation determining the first approximation are prime to each other, and the number of branches in the cycle equals the exponent of $y$. If, however, the exponents of $x$ and $y$ in this equation have a common divisor greater than unity, the corresponding branches break up into cycles equal in number to the common divisor and the number of branches in each cycle is the exponent of $y$ divided by the common divisor.
d) If a group of terms is underscored and the equation formed by equating this group to zero has equal roots, these equal roots must be removed before the branches corresponding to the group can be separated. If this equation is now solved the branches will be separated into single branches and cycles of branches, provided the exponents of $y$ in this equation have no common divisor greater than unity. If, however, there is a common divisor greater than unity the branches corresponding to this group break up into sub-cycles.

## V. Applications in Curve Tracing.

Example 1.-Let it be required to trace the curve represented by the equation

$$
\text { (1) } y^{\prime}-3 x^{4} y-x^{8} y+9 x^{7} y+2 x^{6}-2 x^{18}=0
$$

in the neighborhood of the singular point $(0,0)$.

Collecting terms in like powers of $y$
(2) $y^{3}+\left(-3 x^{4}-x^{6}+9 x^{2}\right) y+\left(2 x^{6}-2 x^{8}\right)=0$.

Since in a first approximation the lowest powers of $x$ in the several coefficients alone count, this equation may be written

$$
\text { (3) } y^{2}-3 x^{4} y+2 x^{6}=0
$$

The application of the method of underscored terms shows that in equation (3) the three terms must be underscored as one term, hence

$$
\text { (4) } y^{3}-3 x^{4} y+2 x^{6}=0 \text {. }
$$

The equation $y^{3}-3 x^{4} y+2 x^{6}=0$ has two roots each equal to $x^{3}$.

Diminishing each root of equation (r) by $x^{2}$, if we write $y=$ $y_{1}+x^{2}$, we obtain the equation,

$$
\text { (5) } y_{1}^{3}+3 x^{2} y_{1}^{2}+\left(-x^{6}+9 x^{2}\right) y_{1}+\left(-3 x^{8}+9 x^{9}\right)=0 \text {. }
$$

Retaining for a first approximation only the lowest powers of $x$

$$
\text { (6) } y_{1}^{3}+3 x^{3} y_{1}{ }^{2}-x^{6} y_{1}-3 x^{8}=0 \text {. }
$$

In equation (6) the terms $\mathbf{I}, 2,4$ must be underscored, that is

$$
\text { (7) } y_{1}^{3}+3 x^{2} y_{1}{ }^{3}-x^{6} y_{1}-3 x^{8}=0 \text {. }
$$

From equation (7) the first approximations of $y_{1}$ are

$$
\text { (8) } y_{1}=-3 x^{2}, y_{1}=x^{3}, y_{1}=-x^{3} \text {. }
$$

Consequently the first approximations of $y$ are

$$
\text { (9) } y=-2 x^{3}, y=x^{2}+x^{3}, y=x^{2}-x^{3} \text {. }
$$

The three branches of $y$ are separated by these approximations and the behavior of the curve at the multiple point is found by tracing the three equations ( 9 ) in the neighborhood of the origin.

Example II.-Let it be required to trace the curve represented by the equation

$$
\text { (1) } y^{3}-3 x^{4} y+9 x^{7} y+2 x^{6}=0
$$

in the neighborhood of the singular point $(\mathrm{o}, \mathrm{\circ})$.

To obtain a first approximation this equation may be written

$$
\text { (2) } y^{3}-3 x^{4} y+2 x=0 \text {. }
$$

The three terms of this equation must be underscored as a single term, when it is found that the equation from which the first approximations are to be found has two roots each equal to $x^{2}$.

Transforming equation (2) by writing $y=y_{1}+x^{2}$, there results

$$
\text { (3) } y_{1}^{3}+3 x^{2} y_{1}^{2}+9 x^{7} y_{1}+9 x^{9}=0 \text {. }
$$

In equation (3) terms $\mathrm{I}, 2,4$ must be underscored, which gives
(4) $y_{1}{ }^{3}+\underline{3 x^{2} y_{1}{ }^{2}}+9 x^{2} y_{1}+9 x^{0}=0$.

The first approximations of $y_{1}$ are

$$
\text { (5) } y_{1}=-3 x^{2}, y_{1}=3 i x^{\frac{3}{2}}, y_{1}=-3 i x^{\frac{3}{3}} \text {, }
$$

and consequently the first approximations of $y$

$$
\text { (6) } y=-2 x^{2}, y=x^{3}+3^{i x^{3}}, y=x^{2}-3^{i x^{\frac{3}{2}}} \text {. }
$$

The approximations (6) separate the three branches of the curve at the multiple point.

## Vi. Applications in Functions of the Complex Variable.

Example I.-Let it be required to determine the behavior of the five-valued algebraic function defined by the equation

$$
\text { (1) } w^{5}-\left(1-z^{2}\right) w^{4}-\frac{4^{4}}{5^{6}} z^{2}\left(1-z^{2}\right)^{4}=0
$$

at the branch-points of the function.
The branch-points, the common solutions of ( 1 ) and the partial derivative of ( 1 ) with respect to $z$,

$$
\text { (2) } 5 w^{4}-4\left(\mathrm{r}-z^{2}\right) w^{5}=0
$$

are located at $z=0, z= \pm 1$.
At $z=0$ the first approximations of $z 0$ are determined by the equation

$$
\text { (3) } w^{6}-w^{4}-\frac{4^{4}}{5^{3}} \frac{z^{2}}{}=0
$$

These first approximations are
1, $a z^{\frac{1}{2}},-a z^{\frac{1}{2}}, a i z^{\frac{1}{2}},-a i z^{\frac{1}{4}}$, where $\alpha$ satisfies the equation

$$
a^{4}=-\frac{4^{4}}{5^{5}}
$$

This shows that at the origin there is one separate branch, and two separate cycles of two branches each.
To determine the behavior of the function at $z= \pm 1$, place $z=$ $z^{\prime} \pm 1$ in equation ( 1 ). There results
(4) $w^{5}-\left(\mp 2 z^{\prime}-z^{2}\right) w^{4}-\frac{4^{4}}{5^{5}}\left(z^{\prime} \mp 1\right)^{2}\left(\mp 2 z^{\prime}-z^{2}\right)^{4}=0$,
which for a first approximation may be written

$$
\text { (5) } w^{5} \pm 2 z^{\prime} w^{4}-\frac{4^{6}}{5^{5}} z^{\prime 4}=0 \text {. }
$$

The first approximations are

$$
w=\frac{4}{5}(4)^{\frac{z}{z}} z^{7}
$$

from which it is seen that at the branch-points $z= \pm 1$ five branches of the function hang together in a cycle.
To determine the behavior of the function at the point $z=\infty$, $w=\infty$, substitute in $(1) z=\frac{1}{z^{\prime}}, w=\frac{1}{w^{\prime \prime}}$, whence

$$
\text { (6) } \frac{4^{4}}{5^{6}}\left(1-z^{\prime 2}\right) w^{4}-z^{\prime \prime}\left(1-z^{\prime 2}\right) w^{\prime}-z^{10}=0 .
$$

Equation (6) for a first approximation at ( $u^{1}=0, z^{\prime}=0$ ) may be written

$$
\text { (7) } \frac{4^{4}}{5^{5}} \underline{w^{6}}-z^{\prime 0} w^{\prime}-z^{\prime 10}=0 \text {. }
$$

Equation (7) has two roots each equal to $-\frac{5}{4} z^{z^{\prime 2}}$. Increasing each of the five roots of (6) by $+\frac{5}{4} z^{2}$ and retaining for a first approximation only the lowest powers of $z^{\prime}$ in the several coefficients,
(8) ${\frac{5^{4}}{} 5^{4}}^{w^{\prime 3}}-\frac{4^{3}}{5^{3}} z^{\prime} w w^{4}+\frac{4^{2}}{5^{2}} z^{\prime 4} w w^{\prime 3}-\frac{4}{5} z^{\prime 0} w \prime^{\prime 2}-z^{\prime 20} w^{\prime}-z^{\prime 10}=0$.

Equation (8) shows that at the point ( $w=\infty, z=\infty$ ) the function has five separate branches, that is the point at infinity is not a branch-point of the algebraic function.

Example II.-To illustrate the method of finding the successive approximations let it be required to determine to three terms the expansion of the branches of the cycle corresponding to the underscored terms of the equation

$$
\text { ( ( ) } w^{5}-z^{4} w^{2}-z^{7} w-z^{10}=0 \text {. }
$$

Introducing a factor $t$ into the terms of ( 1 ) which are not underscored, then differentiating twice with respect to $t$ considering $z$ constant,

$$
\text { (2) } w^{5}-z^{4} w^{3}-z^{7} w t-z^{10} t=0 \text {. }
$$

$$
\text { (3) } 5 w^{4} \frac{d w}{d t}-2 z^{4} w \frac{d z v}{d t}-z^{7} t \frac{d z v}{d t}-z^{7} w-z^{10}=0 .
$$

(4) $\left(5 w^{4}-2 z^{4} w-z^{7} t\right) \frac{d^{2} w}{d t}+\left(20 w^{3}-2 z^{4}\right) \frac{d w^{2}}{d t^{2}}-2 z^{\frac{7}{4}} \frac{d z}{d t}=0$.

Making $t=0$ in (2), (3), (4)

$$
(w)_{0}=z^{\frac{3}{3}},\left(\frac{d z w}{d t}\right)_{0}=\frac{1}{3} z^{3}+\frac{1}{3} z^{\frac{1}{3}},\left(\frac{d^{2} w}{d l^{2}}\right)_{0}=-\frac{2}{9} z^{3^{4}} .
$$

Substituting in Maclaurin's series

$$
w=(w)_{0}+\left(\frac{d w}{d t}\right)_{0} t+\left(\frac{d^{\prime} w^{2}}{d t^{2}}\right)_{0} \frac{t^{2}}{2}+\ldots
$$

and making $t=\mathrm{I}$ in the result we find

$$
\text { (5) } w=z^{3}+\frac{1}{3} z^{3}+\frac{1}{3} z^{13}
$$

which is correct to three terms. Equation (5) has the form of a power series in $z^{\frac{1}{4}}$ beginning with the fourth power and represents a cycle of three branches of the algebraic function whanging together at the singular point.

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